MAT 534: HOMEWORK 13

DUE M Nov 30

Problems marked by asterisk (*) are optional.

Throughout this assignment, F is an arbitrary field.

1. Find all eigenvectors of the operator T on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$T = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}.$$

Is it diagonalizable?

2. Let $T \in \mathrm{End}_F V$ be such that its characteristic polynomial $c_T(x)$ completely decomposes in the field F.

(a) Prove that T is diagonalizable iff there exists a polynomial p without multiple roots such that $p(T) = 0$.

(b) Let T be diagonalizable and let a subspace $W \subseteq V$ be an invariant subspace for T, that is, $T(W) \subseteq W$. Prove that restriction of T to W is also diagonalizable.

(c) Prove that if a square matrix A over $\mathbb{C}$ satisfies $A^k = I$, then it is diagonalizable.

(d) Prove that if a square matrix A over a filed F of characteristic different from 2 satisfies $A^3 = A$, then it is diagonalizable.

***3.** Let $A, B \in \mathrm{End}_F V$ be commuting operators: $AB = BA$.

(a) Show that if $V(\lambda)$ is the generalized eigenspace for A (that is, $V(\lambda) = \ker(A - \lambda I)^k$, where k is the multiplicity of root λ of $c_A(x)$), then $BV(\lambda) \subseteq V(\lambda)$.

(b) Show that if A and B are both diagonalizable, then they can be diagonalized simultaneously: there is a basis of V in which both A and B are diagonal.

(c) Let $C : \mathbb{C}^n \to \mathbb{C}^n$ be an operator defined by

$$C \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{x_n + x_2}{2} \\ \frac{x_1 + x_3}{2} \\ \vdots \\ \frac{x_{n-1} + x_1}{2} \end{bmatrix}.$$

Diagonalize the operator C. (*Hint*: It commutes with the operator of cyclic permutation of x_i).

4. Use Jordan canonical form to prove that if A is a real $n \times n$ matrix such that all its eigenvalues are real and positive, then there exists a real matrix B such that $B^2 = A$.

***5.** Consider the following matrix

$$A = \begin{pmatrix} 9 & 4 & 5 \\ -4 & 0 & -3 \\ -6 & -4 & -2 \end{pmatrix}.$$

(a) Find the rational canonical form for C and determine a matrix P which conjugates C to its rational canonical form.

(b) Using part (a), determine the Jordan canonical form for C and the matrix which conjugates C to its Jordan canonical form.

6. Let V be a finite-dimensional $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} V = n$.

(a) Let $V_{\mathbb{R}}$ be the space V, considered as an $\mathbb{R}$-vector space. Show that $\dim_{\mathbb{R}} V_{\mathbb{R}} = 2n$.

(b) Let $\langle u, v\rangle$ be a Hermitian inner product on V. Prove that $(u, v) = \mathrm{Re}\langle u, v\rangle$ is an inner product on $V_{\mathbb{R}}$, and that the bilinear form $\omega(u, v) = \mathrm{Im}\langle u, v\rangle$ on $V_{\mathbb{R}}$ is alternating.